Transition scattering in stochastically inhomogeneous media

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When a physical object ("a source") without its own eigenfrequency moves through an acoustically homogeneous medium, the only possible form of acoustic radiation is the emission of Mach shock waves, which appear when the source velocity surpasses sonic speed. In nonhomogeneous media, in nonstationary media, or in the neighborhood of such media, the source motion is accompanied by the so-called "transition" radiation (diffraction or scattering), which has place even when the source moves with subsonic velocity. Key features pertaining to the formation of the acoustical transition scattering in media with fluctuating acoustical parameters are established. To analytically study the effect, the Green's function method formulated in terms of functional derivatives is used. The relationship between the wave number and frequency, $k=k(\omega)$, for acoustic waves is found. The results serve to determine the phasing conditions necessary for opening the transition scattering and Cherenkov radiation channel and to establish the physical explanation for the phenomenon—scattering (transformation) on inhomogeneities of the accompanied source field; i.e., formation of radiation appears when the attached field readjusts back to the equilibrium state after being deformed while passing through the fluctuations of the medium.
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I. INTRODUCTION

This paper investigates the phenomenon of transition radiation (scattering) of acoustical waves by an object (not possessing its own proper frequency)[1] uniformly moving in a stochastically nonhomogeneous medium in sub- and supersonic regimes.

Generation of waves caused by the source moving in various types of medium has attracted particular interest because of the wide range and importance of its applications to acoustics, optics, geophysics, and other areas of physics and due to its critical role in the broader theory of wave propagation (see, for example, Refs. 2–5 and references therein). It has been established that if the source moves uniformly at a supersonic speed in a homogeneous medium, it generates what is called "Cherenkov radiation." If the source moves at a constant but subsonic speed, it generates waves if it crosses the interface of two media with different properties ("transition radiation") or if it moves near some boundary ("transition diffraction") (see Ref. 6 and references therein). The source can also radiate if it accelerates.

To outline the geometry of the effect, consider Fig. 1 commonly used to describe Cherenkov radiation.[7,8] Figure 1 illustrates a point source moving from point A toward point B with constant velocity $\mathbf{V}$. At point A the source generates a wave with phase speed in medium c. By the time the source reaches point B at distance Vt, the spherical wave from point A propagates at distance ct.

Next look at line A-D in the direction of wave vector $\mathbf{k}$ of a field spectral component. The phase difference $\Delta\psi(\omega)$ between spherical waves, $\sim\exp(-i\omega t+ikr)/r$, generated at points A and B and observed at "infinity" along angle θ to the trajectory of the body, is given by expression $\Delta\psi=k(DA-CA)=k(Vt\cos\theta-ct)$ because $\psi^B(\omega)=\psi^C(\omega)$. At large distances, we can neglect the difference between $r_{A\infty}$ and $r_{B\infty}$ when considering amplitudes, but for phase relationships this distinction is essential. The waves do not cancel each other at infinity if $\Delta\psi=k(V\cos\theta-c)t\ll\pi$. For any t, this condition is realized *only* for $\cos\theta=c/V$, i.e., when $V>c$. This is the Cherenkov effect.

However, if the medium properties are not uniform, but rather stochastically fluctuating, the phases of waves at points C and B (Fig. 1) do not necessarily have the same values as above. Therefore, the phasing conditions for superposing waves at infinity may not hold. To derive the phase difference leading to radiation requires special calculations incorporating information about the medium.

Figure 2 illustrates the physics behind the formation of the "transition scattering." On the left side of Fig. 2, our source moves uniformly in a homogeneous medium and is accompanied by an attached wave field,[9] distributed in accordance with the properties of the surrounding region. It is precisely this attached wave field and not the source itself (whether a point or nonpoint) that leads to the additional radiation [circle (1) represents the surfaces of equal stress]. As the source travels through region D filled with inhomogeneities, such that the equilibrium parameters of the medium differ from the values describing the homogeneous region, the attached field becomes deformed by the fluctuating parameters within the region [circle (2)]. Past the inhomogeneous region, the attached field begins to rearrange itself toward the configuration corresponding to the equilibrium parameters of the surrounding medium [circle (3)]. Because the energy and source velocity remain constant in this process, the rearrangement of the attached field gives rise to an

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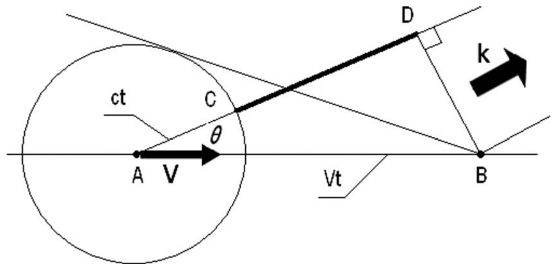

FIG. 1. Phasing condition for Cherenkov radiation.

additional field, namely, the radiation field. The resulting radiation is the transition scattering. Clearly transition scattering can be viewed as a subset of transition (transformation) radiation.

The distinction in the nature of transition scattering and transition radiation can also be seen in the difference of the process durations. Transition scattering is a continuous process caused by the rearrangement of the attached field, and therefore, it lasts effectively infinitely. Transition radiation, on the other hand, has a short-term impulselike character. It occurs when the object transits through the variation in the medium. While extensive literature[7,10] has addressed both transition radiation and transition scattering problems in electrodynamics, acoustical transition scattering has not been investigated to such depths. Problems of acoustic transition radiation have been surveyed in detail by Pavlov and Sukhorukov.[6] The problem of transition scattering in a turbulent medium (i.e., the one where the medium fluctuations are caused by fluctuations in velocities of fluid particles), but not the medium with fluctuating state parameters (such as sound speed, pressure, or density), was considered by Pavlov.[11] The effect of transition scattering near a rough surface was analyzed by Pavlov and Sukhorukov.[12] Lipovskii and Tamoikin[13] studied a specific model for the fluctuating parameter medium using a model relationship between the Fourier transform of the average momentum per unit volume of the medium and the velocity Fourier component. Our current paper aims at establishing a clear and unambiguous description, in terms of the Green's function, for the radiation field created by a moving source in stochastically fluctuating media with varying sound speed. Along with finding the characteristics of radiation by sources moving in fluctuating media, the development of the functional Green's function method produces a general methodology that can be applied to a variety of problems with differing specifics and assumptions.

The remainder of this paper is organized as follows. In Sec. II we define our model and the basic equations and address intensity, energy flux, and energy density relationships in the frequency domain. We propose a simple method (Sec. III) for analyzing wave propagation through a fluctuating medium, for calculating wave characteristics based on the Green's function method with functional derivatives, and for low-magnitude fluctuation approximations. We analytically calculate the dispersion relationship and coefficient of attenuation for the averaged component of the wave (acoustical) field. The results obtained at this step of our analysis serve to establish the phase conditions necessary for the opening of the Cherenkov radiation channel (Sec. IV). The angular-spectral power of the scattering radiation is considered in Sec. V. Section VI summarizes our results. In the medium with strongly fluctuating sound speed, the conditions for Cherenkov radiation can change drastically. The expression, obtained in this paper, shows that in such fluctuating medium the radiation channel opens for the *subsonic* Mach numbers, $M<1$. The shock wave with a *sharp* front does not form in this case because different spectral components are radiated under different angles. The relationship between the radiated angles for short and long waves makes experimental verification possible. This relationship should be taken into consideration when the power of transition radiation is derived.

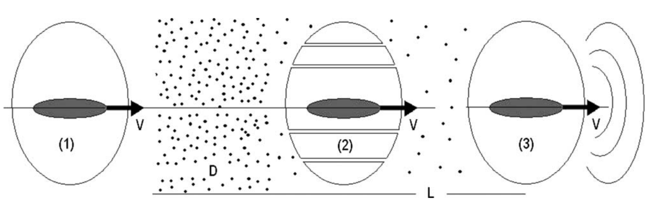

FIG. 2. Formation of transition scattering.

II. BASIC EQUATIONS AND ENERGETIC RELATIONS

As noted above, transition scattering radiation arises when a source moves through a medium whose properties are such that the speed of sound fluctuates. These fluctuations can be caused by a variety of natural phenomena. Frequently, sound speed fluctuations occur due to random variations of density, but such variations are typically small, and the resulting transition scattering effect is rather weak. Only near the phase transition points does the effect become significant. However, sound speed fluctuations may be rather significant in the medium of mixed nature such as when air bubbles are present in water (as in an upper oceanic layer or a jet wake). Then the transition scattering effect becomes much more pronounced. (Appendix A discusses both of the mentioned scenarios in more detail.)

Two factors determine the conditions under which a source moving in a fluctuating medium radiates acoustical waves—the dispersion relationship between wave number and frequency, which is described by the averaged Green's function, and the phasing condition between superposing emitted waves. To study the effect, we will start by considering the Green's function for a simple model (Appendix A). Consider the equation describing a scalar (for example, acoustical) field generated by a localized (point) source,

$$\Delta\phi - \frac{1}{c^2}(1+\epsilon(\mathbf{x}))\partial_{tt}\phi = F(t)\delta(\mathbf{x}-\mathbf{x}_0(t)). \tag{1}$$

Here, positions of the source and the receiver are defined by coordinates $\mathbf{x}_0$ and $\mathbf{x}$, respectively. Operator Δ is the three-dimensional (3D)-Laplacian operator, and $\delta(\mathbf{x}-\mathbf{x}_0)$ is the 3D-Dirac function used to describe the source as a point. If ϕ denotes standard velocity potential, then "observable" physical variables—acoustical pressure and velocity—are described by the rule $p_1=-\rho_0\partial_t\phi$, $\mathbf{v}_1=\nabla\phi$. The source term describes a volume injection term or a thermic source.[6] The productivity of the source is characterized by function $F(t)$ (which is in units of volume per unit time). The local sound speed is defined by the state equation $c_{\rm loc}^2=(\partial p/\partial\rho)$. The wave speed profile $c_{\rm loc}(\mathbf{x})$ contains all information about the medium that is necessary to describe the process. We assume that the speed fluctuates: $c_{\rm loc}^{-2}=c^{-2}(1+\epsilon(\mathbf{x}))$. Quantity $\epsilon(\mathbf{x})$ is an acoustical fluctuating parameter. In principle, it can be any operator. Further, we will disregard temporal dependence of the variables due to fluctuations. The fluctuations are described by the zero average, $\langle\epsilon(\mathbf{x})\rangle=0$, and correlation function, $\langle\epsilon(\mathbf{x})\epsilon(\mathbf{x}')\rangle=B(\mathbf{x},\mathbf{x}')$, such that $B(0)=\langle\epsilon^2\rangle<1$. For a spatially homogeneous medium, the correlation function is a function of only coordinate difference, $B(\mathbf{x},\mathbf{x}')\equiv B(\mathbf{x}-\mathbf{x}')$. The correlation function is characterized by two parameters: the mean square fluctuation ($\langle\epsilon^2\rangle$) and the radius of correlation (l) describing the characteristic distance over which fluctuation correlation vanishes.[2,3] Obviously, more complex models can be constructed.

When multiplied by $\partial_t\phi$ and integrated with respect to volume, Eq. (1) becomes

$$\partial_t E' + \int d\mathbf{x}\,\mathrm{div}\,\mathbf{S}' = -\int d\mathbf{x}\partial_t\phi(\mathbf{x},t)F(t)\delta(\mathbf{x}-\mathbf{x}_0(t)), \tag{2}$$

where

$$E' = \frac{1}{2}\int d\mathbf{x}\left[(\nabla\phi)^2 + \frac{1}{c^2}(1+\epsilon(\mathbf{x}))(\partial_t\phi)^2\right] \tag{3}$$

and $\mathbf{S}'=-\partial_t\phi\nabla\phi$. Here, the field energy includes both the energy of free and attached to the source fields.[14] It also includes the energy of field interaction with fluctuations $E_{\rm int}=(2c^2)^{-1}\int d\mathbf{x}\epsilon(\mathbf{x})(\partial_t\phi)^2$. The term

$$A_r = -\int d\mathbf{x}\partial_t\phi(\mathbf{x},t)F(t)\delta(\mathbf{x}-\mathbf{x}_0(t)) \tag{4}$$

describes the work performed by the moving source against the radiation friction force.[15] Vector $\mathbf{S}'=-\partial_t\phi\nabla\phi$ defines the density of the energy flux. The integral

$$W(t) = \int d\mathbf{x}\,\mathrm{div}\,\mathbf{S} \equiv \oint_\Sigma d\mathbf{f}\cdot\mathbf{S} \tag{5}$$

defines the power of wave radiation. The surface integral is calculated over the "wave zone"—the integrable surface Σ placed at such a distance from the origin of coordinates (where the source moves but does not cross the surface) that the generated field has a structure of divergent spherical waves. For a chosen pulsation ω of a spectral component ϕ_ω, radius r of such a sphere is confined by the condition $c/\omega \ll r \ll \gamma^{-1}$. Here, γ is a dissipative factor (logarithmic decrement) since dissipation always exists in media. The time-averaged work of external forces against radiation damping is compensated by losses on radiation, $\bar{A}_r$, and the reorganization of the attached field of the source. The time averaging is defined by the integral $\overline{A(t)}=\lim_{T\to\infty}T^{-1}\int_0^T dtA(t)$.

Using Fourier transformation with respect to time, we obtain a time-averaged expression for the radiation power,

$$\begin{aligned}\bar{W} &= \lim_{T\to\infty}\frac{1}{T}\int dt\left[-\oint d\mathbf{f}\cdot\partial_t\phi\nabla\phi\right]\\ &= \lim_{T\to\infty}\frac{1}{T}\int\frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}\int dt e^{-i(\omega-\omega')t}\\ &\quad\times\oint d\mathbf{f}\cdot(i\omega\phi_\omega\nabla\phi^*_{\omega'})\\ &= \int_0^{+\infty}d\omega\left[\lim_{T\to\infty}\frac{1}{2\pi T}\oint d\mathbf{f}\cdot(i\omega\phi_\omega\nabla\phi^*_\omega)+\text{c.c.}\right].\end{aligned} \tag{6}$$

At large distances from the origin, the radiated intensity is defined as the amount of energy flowing per unit time through the area element on a spherical surface with radius r and centered at the origin. The surface integral can be transformed into an integral with respect to the solid angle $d\mathbf{f}[\cdots]=r^2d\Omega[\cdots]$, where $d\Omega=\sin\theta d\theta d\varphi$ is a solid-angle element. The angular-spectral density of radiation is defined by the following expression:

$$\bar{W}_{\omega,\mathbf{n}} = \lim_{T\to\infty}\frac{r^2}{T}\left[\frac{i}{2\pi}ck\phi_\omega\partial_r\phi^*_\omega + \text{c.c.}\right], \tag{7}$$

normalized as $\bar{W}=\int_0^\infty d\omega\int d\Omega\bar{W}_{\omega,\mathbf{n}}$. Here, $k=\omega/c$, and $\mathbf{n}$ is the unit vector drawn from the origin to the point of observation in the radiation direction. When averaged with respect to the fluctuation, this expression gives the spectral density of radiation. It is clear that expression (7) describes two processes: Cherenkov and scattering radiations. In fact, the field is decomposed into the regular and fluctuating components, $\phi=\langle\phi\rangle+\phi'$, and, therefore, $\langle|\phi_\omega|^2\rangle=|\langle\phi_\omega\rangle|^2+\langle\phi'_\omega\phi'^*_\omega\rangle$. To simplify formula notations, below we will use the notation $\phi_\omega\equiv\phi_k$ to describe the temporal Fourier transform with $k=\omega/c$. (Do not confuse it with the spatial Fourier transformation!)

Thus, Cherenkov radiation is expressed via the formula

$$\langle\bar{W}_{\omega,\mathbf{n}}\rangle^{\rm Ch} = \lim_{T\to\infty}\frac{r^2}{T}\left[\frac{i}{2\pi}ck\langle\phi_k\rangle\partial_r\langle\phi^*_k\rangle + \text{c.c.}\right], \tag{8}$$

and the scattering radiation is described by

$$\langle\bar{W}_{\omega,\mathbf{n}}\rangle^{\rm sc} = \lim_{T\to\infty}\frac{r^2}{T}\left[\frac{i}{2\pi}ck\langle\phi'_k\partial_r\phi^{*\prime}_k\rangle + \text{c.c.}\right]. \tag{9}$$

All these quantities are calculated in the wave zone, and therefore, radius r has to be chosen to satisfy condition $k^{-1}\ll r\ll\gamma^{-1}$, where γ is the logarithmic decrement. In this case, $\partial_r\phi_k\simeq\Gamma_k\phi_k$. The analytical expressions for Γ and real part $\Re\Gamma$ are calculated in following sections. Equations (8) and (9) become

$$\langle\bar{W}_{\omega,\mathbf{n}}\rangle^{\rm Ch} \simeq \lim_{T\to\infty}\frac{r^2}{T}\left[\frac{1}{\pi}ck(\Re\Gamma_k)|\langle\phi_k\rangle|^2\right] \tag{10}$$

and

$$\langle\bar{W}_{\omega,\mathbf{n}}\rangle^{\rm sc} \simeq \lim_{T\to\infty}\frac{r^2}{T}\left[\frac{1}{\pi}ck(\Re\Gamma_k)\langle|\phi'_k|^2\rangle\right]. \tag{11}$$

III. GREEN'S FUNCTION

A. Averaged Green's function

Applying the Fourier transformation with respect to time to Eq. (1), we obtain a straightforward expression for a spectral component of the field,

$$\Delta \phi_k + k^2(1+\epsilon(\mathbf{x}))\phi_k = \hat{\mathcal{F}}_k[F(t)\delta(\mathbf{x}-\mathbf{x}_0(t))]. \tag{12}$$

Parameter $k=\omega/c$ denotes the wave number, and $\hat{\mathcal{F}}_k[\cdots]=\int dt[\cdots]\exp(+i\omega t)$ is the temporal Fourier transform of the argument. Our preliminary goal is to find the averaged field $\langle\phi_k\rangle$.

From Eq. (1), we find

$$\phi_k(\mathbf{x}) = \int d\mathbf{x}_1 G(\mathbf{x},\mathbf{x}_1)\hat{\mathcal{F}}_k[F(t)\delta(\mathbf{x}_1-\mathbf{x}_0(t))]. \tag{13}$$

Here, $G(\mathbf{x},\mathbf{x}_0)$ is the Green's function that satisfies

$$\Delta G(\mathbf{x},\mathbf{x}') + k^2(1+\epsilon(\mathbf{x}))G(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}'). \tag{14}$$

The field follows from Eq. (13) by averaging with respect to fluctuations ϵ,

$$\langle\phi_k(\mathbf{x})\rangle = \int d\mathbf{x}_1\langle G(\mathbf{x},\mathbf{x}_1)\rangle\hat{\mathcal{F}}_k[F(t)\delta(\mathbf{x}_1-\mathbf{x}_0(t))] = \hat{\mathcal{F}}_k[F(t)\langle G(\mathbf{x},\mathbf{x}_0(t))\rangle]. \tag{15}$$

To find the averaged Green's function $\langle G\rangle$, there exists a number of different methods.[2–4,16] Using the method based on functional derivatives (see Appendix B), we derive

$$\langle G(|\mathbf{r}-\mathbf{r}_0|)\rangle = \int \frac{d\mathbf{q}}{(2\pi)^3}e^{i\mathbf{q}\cdot|\mathbf{r}-\mathbf{r}_0|} \times\left[k^2-q^2-\int d\mathbf{x}\Sigma(\mathbf{x})\exp(-i\mathbf{q}\cdot\mathbf{x})\right]^{-1}. \tag{16}$$

Kernel $\Sigma(\mathbf{x},\mathbf{z})$ contains only irreducible diagrams (Fig. 3, see Appendix B and, for example, Ref. 3, p. 358). For isotropic and homogeneous medium, $\Sigma(\mathbf{x})=\Sigma(r)$, where $r=|\mathbf{x}|$. In a spherical coordinate system, after integrating over the angles, Eq. (16) takes the following form:

$$\langle G(R)\rangle = \frac{1}{i4\pi^2 R}\int_{-\infty}^{+\infty} dq q e^{iqR} \times\left[k^2-q^2-4\pi q^{-1}\int_0^\infty dr r\Sigma(r)\sin qr\right]^{-1}. \tag{17}$$

Further analysis requires knowing the explicit form for $\Sigma(r)$. If we retain only the leading first term of Eq. (B13),[17] which is proportional to $\langle\epsilon^2\rangle$, we obtain

$$\Sigma(r) \simeq k^4 B(r)G_0(r). \tag{18}$$

B. Approximation for low-magnitude fluctuations

The poles of the integrand function in Eq. (17) determine the effective wave number of the mean field. They can be found by solving

$$k^2 - q^2 + \frac{k^4}{q}\int_0^\infty dr B(r)e^{ikr}\sin qr = 0. \tag{19}$$

For small $\langle\epsilon^2\rangle$, the equation is solved by iterations. In the zero-order approximation with respect to $\langle\epsilon^2\rangle$, we have $q^{(0)}=k$. The next-order approximation is found from

$$k^2 - q^{(1)2} + k^3\int_0^\infty dr B(r)e^{ikr}\sin kr = 0, \tag{20}$$

which gives

$$\Gamma \equiv q^{(1)} = k\left[1+\frac{k}{4}\int_0^\infty dr B(r)\sin 2kr + i\frac{k}{2}\int_0^\infty dr B(r)\sin^2 kr\right] + \cdots. \tag{21}$$

Here, we used $(1+2x)^{1/2}\simeq 1+x+\cdots$ for small x. For practical applications, it is convenient to use Fourier transforms of the correlation function: $B(\mathbf{s})=\int d\mathbf{x}B(r)e^{-i\mathbf{s}\cdot\mathbf{x}}$. After simple calculations in spherical coordinates, we find that $B(\mathbf{s})\equiv B(s)$,

$$B(s) = \frac{4\pi}{s}\int_0^\infty dr r B(r)\sin sr \rightleftharpoons B(r) = \frac{1}{2\pi^2 r}\int_0^\infty ds s B(s)\sin sr. \tag{22}$$

By substituting $B(r)$ from Eq. (22) into Eq. (19), we obtain expressions where the following integrals are present: $I_1=\int_0^\infty dr r^{-1}\sin sr\sin 2kr$, $I_2=\int_0^\infty dr r^{-1}\sin sr\sin^2 kr$. These integrals indeed *can* be calculated analytically, and their calculation presents some methodological interest. Let us use the fact that $\int_0^\infty dr\exp(i\xi-\alpha)r=(\alpha-i\xi)^{-1}$ for $\alpha>0$. Consider $\alpha\to+0$. In this case, we can write that

$$\int_0^\infty dr e^{i(2k\pm s)r} = \frac{1}{+0-i(2k\pm s)} \to$$

$$\int_0^\infty dr\sin sr e^{i2kr} = \frac{1}{2i}\left[\frac{1}{+0-i(2k+s)}-\frac{1}{+0-i(2k-s)}\right].$$

By separating the real and imaginary parts of the integral, and calculating with respect to parameter k, we find

$$\int_0^\infty \frac{dr}{r}\sin sr\sin 2kr = \Re\frac{1}{2}\ln\left[\frac{(+0-i(2k+s))}{(+0-i(2k-s))}\frac{(+0+is)}{(+0-is)}\right] = \frac{1}{2}\ln\left|\frac{(2k+s)}{(s-2k)}\right|,$$

$$\int_0^\infty \frac{dr}{r}\sin sr\sin^2 kr = \Im\frac{1}{4}\ln\left[\frac{(+0-i(2k+s))}{(+0-i(2k-s))}\frac{(+0+is)}{(+0-is)}\right] = -\frac{1}{4}\arg(2k-s) = \frac{\pi}{4}H(2k-s). \tag{23}$$

Here, $H(z)$ is the Heaviside function: $H(z)=1$ for $z>0$ and $H(z)=0$ for $z<0$. We used the fact that $\ln z=\ln|z|+i\arg z$ and $\arg(s-2k)=-\pi$ for $s<2k$ because the branch point is at $s=2k+i0$, which we must contour *clockwise*. By substituting

Eq. (22) into Eq. (19) and using Eq. (23), we find that expression (19) takes form

$$\begin{aligned}\Gamma &\equiv K_k + i\gamma_k \\ &\equiv k[1+\kappa_k + i\delta_k] \\ &= k\left[1 + \frac{k}{8\pi^2}\frac{1}{2}\int_0^\infty dssB(s)\ln\left|\frac{2k+s}{2k-s}\right| \right. \\ &\quad \left. + i\frac{k}{4\pi^2}\frac{\pi}{4}\int_0^{2k} dssB(s)\right].\end{aligned} \tag{24}$$

C. Dispersion relationship and coefficient of attenuation

By combining Eqs. (24) and (17), we find that the averaged Green's function is approximated by

$$\begin{aligned}\langle G(\mathbf{x}-\mathbf{z})\rangle &\simeq -\frac{1}{4\pi}e^{-\gamma_k|\mathbf{x}-\mathbf{z}|}\left.\frac{\exp(+iK_k|\mathbf{x}-\mathbf{z}|)}{|\mathbf{r}-\mathbf{z}|}\right|_{r\gg|\mathbf{z}|} \\ &\to -\frac{1}{4\pi}e^{-\gamma_k r}\frac{\exp(+iK_k r)}{r}\exp(-iK_k\mathbf{n}\cdot\mathbf{z}).\end{aligned} \tag{25}$$

The dispersion relationship between the wave number and frequency and the attenuation coefficient of the averaged field amplitude are given by

$$K_k = k\left[1 + \frac{k}{8\pi^2}\frac{1}{2}\int_0^\infty dssB(s)\ln\left|\frac{2k+s}{2k-s}\right|\right],$$

$$\gamma_k = \frac{1}{16\pi}k^2\int_0^{2k} dssB(s). \tag{26}$$

Here, $k=\omega/c$. Notice that if there is dissipation, then there is also dispersion: different spectral components of the averaged field propagate with different phase speeds. Subsequent calculations require knowing the exact structure of the correlation function. However, we do know some general properties of this function and can obtain certain insights without such precise expressions. Any correlation function $B(r)$ has a maximum at $r=0$, vanishes when $r\to\infty$, and is characterized by two parameters: the mean square fluctuation, $\langle\epsilon^2\rangle$, and the correlation radius, l, describing the characteristic distance over which fluctuations are no more correlated. As an example, consider $B(s)=\pi^{3/2}\langle\epsilon^2\rangle l^3\exp(-s^2l^2/4)$. This expression corresponds to the correlation function $B(r)=\langle\epsilon^2\rangle\exp(-r^2/l^2)$ commonly used in many practical situations.[18] Simple calculations lead to the following expression for the attenuation coefficient:

$$\gamma_k = \frac{\sqrt{\pi}}{8}\langle\epsilon^2\rangle lk^2(1-\exp(-k^2l^2)). \tag{27}$$

Its limit cases are $\gamma_k=(\sqrt{\pi}/8)\langle\epsilon^2\rangle l^3k^4$ for $k^2l^2\ll 1$ and $\gamma_k=(\sqrt{\pi}/8)\langle\epsilon^2\rangle lk^2$ for $k^2l^2\gg 1$. In this case, the dispersion relationship (wave number K_k of the propagating wave expressed as a function of pulsation $k(=\omega/c)$) has the following form:

$$\begin{aligned}K_k &= k + \frac{k^2}{8\pi^2}\frac{1}{2}\pi^{3/2}\langle\epsilon^2\rangle l^3\int_0^\infty dsse^{-s^2l^2/4}\ln\left|\frac{2k+s}{2k-s}\right| \\ &= k + \frac{k^4}{4\sqrt{\pi}}\langle\epsilon^2\rangle l^3\int_0^\infty dxxe^{-(kl)^2x^2}\ln\left|\frac{1+x}{1-x}\right|.\end{aligned} \tag{28}$$

This integral can be calculated analytically. However, its expression in terms of special functions is very cumbersome. For this reason, we will analyze the behavior of this function only in two limit cases—for long and short waves—and will derive approximate interpolation expressions.

Consider $kl\ll 1$ (long waves). The principal contribution to the integral in Eq. (28) comes from the interval $0<x<(kl)^{-1}$ because exponent $e^{-(kl)^2x^2}$ is a rapidly decreasing function for $x>(kl)^{-1}$. For this reason, we present the integral in Eq. (28) as a sum of two integrals: $\int_0^\infty dx\cdots=\int_0^M dx\cdots+\int_M^\infty dx\cdots\equiv I_1+I_2=I$, where parameter M satisfies the condition $1\ll M<(kl)^{-1}$. In the first integral, I_1, we can replace the exponential $e^{-(kl)^2x^2}$ with 1. In the second integral, I_2, where $x>M\gg 1$, we can write $\ln|(1+x)/(1-x)|\simeq 2/x$. After this simplification both integrals can be calculated analytically. In fact, the first integral gives[19] $I_1=M+\frac{1}{2}(M^2-1)\ln[(M+1)/|M-1|)]\simeq 2M-(2/3M)$. The second gives $2\int_M^\infty dxe^{-(kl)^2x^2}\simeq e^{-(kl)^2M^2}/(kl)^2M\simeq 1/(kl)^2M$. The sum I_1+I_2 is a weakly dependent function of M when $\partial_M I=0$. This helps find $M=(\sqrt{2}kl)^{-1}\gg 1$. Finally, collecting the results of calculations leads to $K_k\simeq k+a_1\langle\epsilon^2\rangle l^2k^3$, which differs drastically from the linear dependence $K_k=k\equiv\omega/c$ existing for a nonfluctuating medium. Here, $a_1=3/(4\sqrt{2\pi})$ is a numerical constant.

If $kl\gg 1$ (short waves), the integral is approximately evaluated as $\simeq(kl)^{-3}$. Indeed, the principal contribution comes from the domain $0<x<(kl)^{-1}\ll 1$, where $\ln|(1+x)/(1-x)|\simeq 2x$. The integral is $2\int dxx^2\exp(-(kl)^2x^2)=\frac{\sqrt{\pi}}{2}(kl)^{-3}$ (see Ref. 20). By collecting the coefficients, we find that the dispersion relationship is written as $K_k=k[1+a_2\langle\epsilon^2\rangle]$, where $a_2=1/8$ is a numerical constant. The obtained expression shows that the averaged component of an acoustical wave propagates in a fluctuating medium with smaller speed, $c_{\mathrm{ph}}=c/[1+a_2\langle\epsilon^2\rangle]<c$, than if the medium does not fluctuate. This effect can be interpreted as scattering and superposition of multiple waves interacting with inhomogeneities.

A close approximation covering both limit cases can be obtained by the following interpolation of Eq. (28):

$$K_k \equiv k(1+\delta_k) = k\left[1 + \langle\epsilon^2\rangle\frac{a_1a_2(kl)^2}{a_2+a_1(kl)^2}\right]. \tag{29}$$

The phase speed of a regular component of a scalar field is defined thus from the approximate expression $c^{-1}(k)=c^{-1}[1+\langle\epsilon^2\rangle a_1a_2(kl)^2/(a_2+a_1(kl)^2)]$.

IV. CHERENKOV RADIATION

Let us now use the obtained results to establish the key conditions for the appearance of Cherenkov radiation for a general case with an inhomogeneous medium. Consider radiation by a point source with constant productivity F_0 traveling at constant velocity $V=cM$ (M is the Mach number) along the x-axis. In this case, the right-hand side of Eq. (1) can be written as $V^{-1}F_0\delta_\perp(\mathbf{x}_\perp)\exp(iM^{-1}kx)$; i.e., the basic equation has the form

$$\Delta\phi_k + k^2(1+\epsilon(\mathbf{x}))\phi_k = \frac{1}{Mc}F_0\delta_\perp(\mathbf{x}_\perp)e^{iM^{-1}kx}. \tag{30}$$

Combining Eqs. (15) and (25), we find that the averaged field component is proportional to the Dirac function,

$$\begin{aligned}\langle\phi_k(\mathbf{x})\rangle &\simeq -\frac{F_0}{4\pi Mc}e^{-\gamma_k r}\frac{e^{+iK_k r}}{r}\int dx e^{i(-K_k\cos\theta+M^{-1}k)x} \\ &\simeq -\frac{F_0}{2Mc}\frac{e^{+iK_k r}}{r}\delta(-K_k\cos\theta+M^{-1}k),\end{aligned} \tag{31}$$

since $\int_{-\infty}^{\infty}dx\exp isx = 2\pi\delta(s)$. Here, θ is the direction of radiation defined by $\mathbf{K}_k\cdot\mathbf{x}=K_k x\cos\theta$. The presence of the delta function in the right side of the expression indicates that wave radiation takes place only if the Dirac function argument is zero. This requirement determines the phasing condition for possible radiation directions in a fluctuating medium, $V\cos\theta_k = c(k)$, resulting in

$$\cos\theta_k = \frac{c}{V}\left[1+\langle\epsilon^2\rangle\frac{a_1a_2(kl)^2}{a_2+a_1(kl)^2}\right]^{-1}; \tag{32}$$

i.e., spectral components with different frequencies radiate at different angles. In the limit case when the medium is uniform and has no fluctuations, i.e., when $\langle\epsilon^2\rangle\to 0$, we naturally obtain the classical result: $\cos\theta_k = M^{-1}$ for any spectral components.

The angular-spectral power of Cherenkov radiation is calculated from Eq. (8),

$$\langle\bar{W}_{\omega,\mathbf{n}}\rangle^{\mathrm{Ch}} = \lim_{T\to\infty}\frac{r^2}{T}\left[\frac{1}{\pi}ckK_k|\langle\phi_k\rangle|^2\right], \tag{33}$$

where Eq. (31) is used. By computing, we obtain the expression that is proportional to $\lim_{T\to\infty}(1/T)\delta^2(a)$. Here, $\delta^2(a)$ is the square of the delta function. Following Landau and Lifshitz[21], we can rewrite this expression as $\delta^2(a)=\delta(a)\times(2\pi)^{-1}\lim_{T\to\infty}\int_{-T/2}^{+T/2}dte^{iat}$ by decomposing one of the delta functions into the Fourier integral. Because of the presence of the delta function, the argument in the exponential can be written as zero; i.e., the exponential becomes replaced by 1. Thus, $\lim_{T\to\infty}(1/T)\delta^2(a)=\lim_{T\to\infty}(1/T)(T/2\pi)\delta(a)$, and the infinite time disappears. This result has the simple physical meaning: if a body travels infinitely long, it radiates the infinite amount of energy, but the energy radiated per unit of time (power) is obviously finite and physically meaningful.

We obtain the simple expression

$$\begin{aligned}\langle\bar{W}_{\omega,\mathbf{n}}\rangle^{\mathrm{Ch}} &= \frac{1}{2(2\pi)^2M}kK_k|F_0|^2\delta(-K_k\cos\theta+M^{-1}k) \\ &= \frac{1}{2(2\pi)^2M}k|F_0|^2\delta\left(\cos\theta-\frac{k}{MK_k}\right)\end{aligned} \tag{34}$$

if it is remembered that K_k is given by Eq. (29) and $\delta^2(s)=\lim_{L\to\infty}(2\pi)^{-1}L\delta(s)$, with $L=VT=McT$. Integrated with respect to the solid angle $d\Omega=d\varphi d\theta\sin\theta$, this expression gives

$$\langle\bar{W}_\omega\rangle^{\mathrm{Ch}} = \frac{1}{4\pi M}k|F_0|^2H\left(M-\frac{k}{K_k}\right). \tag{35}$$

V. TRANSITION SCATTERING

To find the angular-spectral power of scattering radiation, it is necessary to derive an expression for the fluctuating component of the field. Using Eqs. (13)–(15), we obtain that at great distances from the source

$$\begin{aligned}\phi_k'(\mathbf{x}) &= \hat{\mathcal{F}}_k[F(t)G'(\mathbf{x},\mathbf{x}_0(t))] \\ &\simeq \int d\mathbf{y}G_0(\mathbf{x}-\mathbf{y})[-k^2\epsilon(\mathbf{y})]\hat{\mathcal{F}}_k[F(t)\langle G(\mathbf{y},\mathbf{x}_0(t))\rangle] \\ &\simeq \frac{k^2}{4\pi r}e^{ikr}\int d\mathbf{y}e^{-ik\mathbf{n}\cdot\mathbf{y}}\epsilon(\mathbf{y})\hat{\mathcal{F}}_k[F(t)\langle G(\mathbf{y},\mathbf{x}_0(t))\rangle].\end{aligned} \tag{36}$$

Here, $\mathbf{n}=\mathbf{x}/|\mathbf{x}|=\mathbf{n}_\parallel\cos\theta+\mathbf{n}_\perp\sin\theta$, where $\mathbf{n}_\parallel$ is a unit vector in direction $\mathbf{V}$.

Consider a source with constant productivity $(F(t)=F_0)$ moving with constant velocity $(\mathbf{x}_0(t)=\mathbf{V}t)$. In this case,

$$\begin{aligned}\hat{\mathcal{F}}_k[F(t)\langle G(\mathbf{y},\mathbf{x}_0(t))\rangle] &= -\frac{F_0}{4\pi}\int dt\frac{e^{+i\omega t+i\Gamma|\mathbf{y}-\mathbf{x}_0(t)}}{|\mathbf{y}-\mathbf{x}_0(t)|} \\ &= -\frac{F_0}{4\pi V}e^{ikM^{-1}y_\parallel} \\ &\quad\times\int_{-\infty}^{+\infty}ds\frac{e^{-ikM^{-1}s+i\Gamma\sqrt{y_\perp^2+s^2}}}{\sqrt{y_\perp^2+s^2}}.\end{aligned} \tag{37}$$

By replacing $s\to|y_\perp|\sinh s$, we first reduce the integral to the form

$$\begin{aligned}\hat{\mathcal{F}}_k[F(t)\langle G(\mathbf{y},\mathbf{x}_0(t))\rangle] &= -\frac{F_0}{4\pi V}e^{ikM^{-1}y_\parallel}\times\int_{-\infty}^{+\infty}ds\exp[-ikM^{-1} \\ &\quad\times|y_\perp|\sinh s + i\Gamma|y_\perp|\cosh s].\end{aligned} \tag{38}$$

Next, by introducing $\Gamma|y_\perp|=\cosh\alpha, kM^{-1}|y_\perp|=\sinh\alpha$, we reduce the integral to the expression that is proportional to the Sommerfeld integral[22] $Z_0(z)$,

$$\begin{aligned}&\hat{\mathcal{F}}_k[F(t)\langle G(\mathbf{y},\mathbf{x}_0(t))\rangle] \\ &\quad= -i\frac{F_0}{4V}e^{ikM^{-1}y_\parallel} \\ &\qquad\times\frac{1}{i\pi}\int_{-\infty}^{+\infty}ds\exp[i|\mathbf{y}_\perp|\sqrt{\Gamma^2-k^2M^{-2}}\cosh s] \\ &\quad= -i\frac{F_0}{4V}e^{ikM^{-1}y_\parallel}Z_0(\beta|\mathbf{y}_\perp|).\end{aligned} \tag{39}$$

Integral $Z_0(z)$ with a complex argument z describes the field around the moving source. It coincides with the Hankel function $H^{(1)}(z)$ if the contour of integration is chosen in a special manner (to ensure correct asymptotic conditions for radiation). The Hankel function can obviously be expressed in terms of the MacDonald function.[23] Parameter $\beta = \sqrt{\Gamma^2 - k^2 M^{-2}}$ is a complex parameter, where $\Gamma = K_k + i\gamma_k$ (see Eq. (24)). After all calculations, we find that at far distances ($r \gg L$, where L is the distance traveled by the source)

$$\phi_k'(\mathbf{x}) \simeq \frac{e^{ikr}}{r}\left[-i\frac{cF_0}{4M}\frac{k^2}{4\pi} \right] \times \int d\mathbf{y}_\perp dy_\parallel \epsilon(\mathbf{y}_\perp, y_\parallel) e^{+ik(M^{-1}-\cos\,\theta)y_\parallel} \times e^{-ik\,\sin\,\theta \mathbf{n}_\perp\cdot\mathbf{y}_\perp} Z_0(\beta|\mathbf{y}_\perp|). \tag{40}$$

This expression has the form of a divergent spherical wave $[\phi_k'(\mathbf{x}) \sim r^{-1}e^{ikr}]$, i.e., it truly describes propagating radiation. However, the question of what exactly is the origin of this radiation—the moving source or the fluctuation of the medium—is not an accurate one. Both are the necessary components to form the radiation. If any of them is removed ($F_0, M \to 0$ or $\epsilon \to 0$), the radiation disappears.

By substituting Eq. (40) into Eq. (9) and taking into consideration that $\partial_r \phi_k' \simeq ik\phi_k'$ in the domain $K_k^{-1} \ll r \ll \gamma^{-1}$, the angular-spectral power of the radiation can be calculated from

$$\langle \bar{W}_{\omega,\mathbf{n}} \rangle^{\mathrm{tr}} = \lim_{T\to\infty} \frac{r^2}{T}\left[\frac{1}{\pi} ck^2 \langle \phi_k' \phi_k'^{*} \rangle \right]. \tag{41}$$

To analytically evaluate the expression, we can consider the simplest case of Gaussian distribution: $\langle \epsilon(\mathbf{y}')\epsilon(\mathbf{y}'')\rangle = \langle \epsilon^2 \rangle \exp[l^{-2}(y_\parallel' - y_\parallel'')^2 + l^{-2}(\mathbf{y}_\perp' - \mathbf{y}_\perp'')^2]$. By changing variables $\mathbf{y}', \mathbf{y}'' \to \mathbf{y} = \mathbf{y}' - \mathbf{y}''$, $\mathbf{Y} = \frac{1}{2}(\mathbf{y}' + \mathbf{y}'')$ to calculate the double integral with respect to $y_\parallel'$ and $y_\parallel''$. We obtain

$$\begin{aligned}\lim_{T\to\infty} \frac{r^2}{T}\langle |\phi_k'|^2 \rangle &= \lim_{T\to\infty} \frac{1}{T}\left[-\frac{F_0}{4V}\frac{k^2}{4\pi} \right]^2 \langle \epsilon^2 \rangle \int dy_\parallel' dy_\parallel'' \exp\left[\frac{(y_\parallel' - y_\parallel'')^2}{l^2} + ik\left(\frac{1}{M} - \cos\,\theta \right)(y_\parallel' - y_\parallel'') \right] \\ &\quad \times \int d\mathbf{y}_\perp' d\mathbf{y}_\perp'' e^{-ik\,\sin\,\theta \mathbf{n}_\perp\cdot(\mathbf{y}_\perp' - \mathbf{y}_\perp'') - l^{-2}(\mathbf{y}_\perp' - \mathbf{y}_\perp'')^2} Z_0(\beta|\mathbf{y}_\perp'|) Z_0^*(\beta|\mathbf{y}_\perp''|) \\ &= \lim_{T\to\infty} \frac{L}{VT}\left[-\frac{F_0}{4}\frac{k^2}{4\pi} \right]^2 \langle \epsilon^2 \rangle l\sqrt{\pi} \exp\left[-\frac{1}{4}k^2 l^2 \left(\frac{1}{M} - \cos\,\theta \right)^2 \right] \int d\mathbf{y}_\perp e^{-ik\,\sin\,\theta \mathbf{n}_\perp\cdot\mathbf{y}_\perp - l^{-2}(\mathbf{y}_\perp)^2} \\ &\quad \times \int d\mathbf{Y}_\perp Z_0\left(\beta\left| \mathbf{Y}_\perp + \frac{1}{2}\mathbf{y}_\perp \right| \right) Z_0^*\left(\beta\left| \mathbf{Y}_\perp - \frac{1}{2}\mathbf{y}_\perp \right| \right).\end{aligned} \tag{42}$$

In terms of the modified variables, differentials $d\mathbf{y}'d\mathbf{y}'' = d\mathbf{y}d\mathbf{Y}$. By definition of quantities L and T, the ratio $\lim_{T\to\infty} L/VT = 1$. In Eq. (42) we calculate the integral with respect to $\mathbf{Y}_\perp$ using the theorem of cylindrical function composition.[22] Let us define the angles for integration from the fixed vector $\mathbf{y}_\perp$. The integral is then written as

$$\begin{aligned}\mathrm{G}(y_\perp) &\equiv \int d\mathbf{Y}_\perp Z_0\left(\beta\left| \mathbf{Y}_\perp + \frac{1}{2}\mathbf{y}_\perp \right| \right) Z_0^*\left(\beta\left| \mathbf{Y}_\perp - \frac{1}{2}\mathbf{y}_\perp \right| \right) \\ &= \int_0^\infty dY_\perp Y_\perp \int_0^{2\pi} d\phi \left[H(Y_\perp - y_\perp)\left(\sum_{m=-\infty}^{+\infty} Z_m(\beta Y_\perp) J_m\left(\frac{1}{2}\beta y_\perp \right) e^{im\phi} \sum_{n=-\infty}^{+\infty} Z_n^*(\beta Y_\perp) J_n^*\left(\frac{1}{2}\beta y_\perp \right) e^{-in(\phi+\pi)} \right) \right. \\ &\quad \left. + H(y_\perp - Y_\perp)\left(\sum_{m=-\infty}^{+\infty} J_m(\beta Y_\perp) Z_m\left(\frac{1}{2}\beta y_\perp \right) e^{im\phi} \sum_{n=-\infty}^{+\infty} J_n^*(\beta Y_\perp) Z_n^*\left(\frac{1}{2}\beta y_\perp \right) e^{-in(\phi+\pi)} \right) \right].\end{aligned}$$

Here, the Heaviside step function $H(z)$ is used, $Z_m(x)$ and $Z_n(x)$ are cylindrical functions,[22] and $J_m(x)$ and $J_n(x)$ are the Bessel functions. After integrating with respect to angle ϕ, only diagonal terms remain in the double sum, and we obtain

$$\begin{aligned}\mathrm{G}(y_\perp) &= 2\pi \sum_{m=-\infty}^{+\infty} (-1)^m \int_0^\infty dY_\perp Y_\perp \\ &\quad \times \left[H(Y_\perp - y_\perp)|Z_m(\beta Y_\perp)|^2 \left| J_m\left(\frac{1}{2}\beta y_\perp \right) \right|^2 \right. \\ &\quad \left. + H(y_\perp - Y_\perp)|J_m(\beta Y_\perp)|^2 \left| Z_m\left(\frac{1}{2}\beta y_\perp \right) \right|^2 \right].\end{aligned} \tag{43}$$

It is convenient to rewrite this expression in the form

$$\begin{aligned}\mathrm{G}(y_\perp) &= \frac{2\pi}{|\beta|^2} \sum_{m=-\infty}^{+\infty} (-1)^m \\ &\quad \times \left[\left| J_m\left(\frac{1}{2}\beta y_\perp \right) \right|^2 \int_{|\beta| y_\perp}^\infty ds s \left| \beta Z_m\left(\frac{\beta}{|\beta|}s \right) \right|^2 \right. \\ &\quad \left. + \left| Z_m\left(\frac{1}{2}\beta y_\perp \right) \right|^2 \int_0^{|\beta| y_\perp} ds s \left| \beta J_m\left(\frac{\beta}{|\beta|}s \right) \right|^2 \right].\end{aligned} \tag{44}$$

The principal contribution in the integral with respect to $\mathbf{y}_\perp$ [see Eq. (42)] comes from domain $|\mathbf{y}_\perp| \leqslant l$. We can simplify Eq. (44) for a special case of source motion when Mach number $M \sim 1$. For $M \sim 1$, the effect of transition radiation is very distinctly expressed. Since in this case $|\beta|^{-1} \gg l$, the corresponding integral becomes equal to $\mathrm{G}(y_\perp) \simeq 2\pi C|\beta|^{-2}$. Here, C is a coefficient independent of the source velocity. In fact, the integral with respect to s for $|\beta| y_\perp \to 0$ does not contain any parameters and, for this reason, produces a numerical value of order unity. We can write that

$$\int d\mathbf{y}'_\perp d\mathbf{y}''_\perp e^{-ik \sin \theta \mathbf{n}_\perp \cdot (\mathbf{y}'_\perp - \mathbf{y}''_\perp) - l^{-2}(\mathbf{y}'_\perp - \mathbf{y}''_\perp)^2} \times Z_0(\beta|\mathbf{y}'_\perp|) Z_0^*(\beta|\mathbf{y}''_\perp|) \simeq 2\pi^2 C l^2 e^{-(1/4)k^2 l^2 \sin^2 \theta} |\beta|^{-2}, \tag{45}$$

where we assume that the source velocity is comparable to the sound speed; i.e., β is small. Under this condition, $\beta \simeq ikM^{-1}\sqrt{1 - M^2 - 2\kappa_k - i2\delta_k} \simeq ik\sqrt{2}\sqrt{(1 - M - \kappa_k) - i\delta_k}$.

By combining the obtained results, Eqs. (41)–(45), and taking into account that $\langle \epsilon^2 \rangle < 1$, we find the following expression for the angular-spectral density of transition radiation:

$$\langle \bar{W}_{\omega,\mathbf{n}} \rangle^{\mathrm{tr}} \simeq \frac{2\pi\sqrt{\pi}C}{(8\pi)^2} \frac{c|F_0|^2 \langle \epsilon^2 \rangle l^3 k^4 M^2}{\sqrt{(1 - M^2 - 2\kappa_k)^2 + 4\gamma_k^2}} \times \exp\left[-\frac{1}{4}(kl)^2 \left[\left(\frac{1}{M} - \cos\theta \right)^2 + \sin^2\theta \right] \right]. \tag{46}$$

This expression produces an estimate of the scattering energy output for a subsonic, $M \leqslant 1$, motion in a nonhomogeneous medium. If $M \geqslant 1$, an additional (Cherenkov) channel [Eq. (34)] opens.

VI. CONCLUSION

When a physical object without its own eigenfrequency moves through an acoustically *homogeneous* medium, the only possible form of acoustic radiation is the emission of Mach shock waves, which appear when source velocity surpasses the speed of sound, i.e., when $M_* \equiv M \cos\theta \geqslant 1$ ($M = V/c$ is the Mach number.) In *inhomogeneous* media, in nonstationary media, or in the neighborhood of such media, the motion of the source is accompanied by the so-called transition radiation (scattering and diffraction), which takes place even when the source moves with subsonic velocity.[6]

In the considered case of a strongly fluctuating medium, modeled by Eq. (1), the conditions for Cherenkov radiation can change drastically. In fact, the condition $(kl)^2 = (a_2/a_1) \times (M_*^{-1} - 1)/(1 + a_2\langle \epsilon^2 \rangle - M_*^{-1}) > 0$ [following from Eq. (32)] shows that in such a fluctuating medium the radiation channel opens for the subsonic Mach numbers, $M < 1$. This type of radiation is possible in the framework of our simple model when $(1 + a_2\langle \epsilon^2 \rangle)^{-1} < M \cos\theta_k < 1$.

The shock wave with a sharp front does not form in this case because different spectral components are radiated under different angles. In fact, the relationship between the radiated short and long wave angles is described by a simple formula [Eq. (32)] that makes experimental verification possible.

The characteristics of the transition scattering energy follow from Eq. (46). The power of the radiation tends to zero for both small and large values of wave numbers. The direction of transition radiation depends strongly on the source velocity. As expected, the transition radiation effect disappears when the source velocity approaches zero.

When the source moves at a subsonic speed, the characteristic space scale of the domain (where the source moves and from where the radiation is emitted) must be sufficiently large. This is very important when conducting experimental observations. In fact, the attached field reorganization does not occur instantaneously and takes some time to develop because the relief of stress always occurs with a finite (sonic) velocity. However, during this time the source travels additional distance (L). Therefore, it is necessary that this characteristic scale L is greater than the characteristic distance L_{ph} at which the radiation is formed, $L \gg L_{\mathrm{ph}} \sim \omega^{-1}(V^{-1} - c^{-1})^{-1}$. In fact, the resulting field contains two components: the first ("attached field") describes the intrinsic field of the source, $\phi_k^i \sim \exp(-ikM^{-1}x)$, while the second ("radiated field") is the result of the interaction with inhomogeneities, $\phi_k^r \sim \exp(-i\mathbf{k} \cdot \mathbf{x})$. Thus, the expression for the spectral energy (being a quadratical functional of fields) contains an interference term proportional to the product of these components. This (spatially oscillating) interference term vanishes after the inevitable space, temporary or statistical averaging. This is important to remember when defining the full energy or performing experimental measurements. Therefore, at large distances ($L \gg L_{\mathrm{ph}}$), when the intrinsic field of the source and the free field separate, the radiation can be observed and registered.

The presented analysis of wave propagation and radiation by moving sources in the fluctuating media was made with the assumption that the fluctuation level is not too high: $(kl)^4\langle \epsilon^2 \rangle < 1$. For this reason, we were able to limit our consideration only to the first term in Eq. (B13). However, for liquids, in the vicinity of the critical point[20] where acoustical parameters of the medium exhibit large fluctuations, the subsequent expansion terms may be needed in Eq. (B13). Obviously, in general, it is not possible to calculate all diagrams, and, therefore, some diagrams have to be omitted. Under certain conditions, however, it is feasible to take all higher-order contributions into account without much extra effort. For example, if we consider kernel $\Sigma(r) \simeq k^4 G_0(r) B(r) \to \Sigma(r) \simeq k^4 \langle G(r) \rangle B(r)$, we can obtain the so-called *self-consistent* approximation. In this context, only the last diagram in the series for Σ of Fig. 3(b) is needed. Physically the self-consistent approximation is very natural: it describes the signal propagation from one fluctuation (scatterer) to another, which happens not in empty space but in space filled with other scatterers.

We considered the basic features of the effect of transition scattering in the framework of the simplest scalar (acoustical) model, Eq. (1). However, the transition radiation and transition scattering are universal physical phenomena.

They might exist not only for scalar (acoustical) fields[6,12,24] but also for other types of waves of arbitrary physical nature.[7]

APPENDIX A: ACOUSTIC PARAMETER FLUCTUATIONS IN MEDIA

As noted above, transition scattering radiation arises when a source moves through a medium whose properties are such that the speed of sound fluctuates due to a variety of natural phenomena. Such hydrodynamical systems are described by equations of the type of Eq. (13)–(15). Below we consider two simplest examples.

The first example is the system with inhomogeneous distribution of density. The mass and momentum conservation equations are $D_t\rho+\rho\,\mathrm{div}\,\mathbf{v}=0$ and $\rho D_t\mathbf{v}+\nabla p=\mathbf{f}$. The full derivative with respect to time is described by the differential operator $D_t=\partial_t+(\mathbf{v}\cdot\nabla)$. Other notations are standard: ρ is density, p is pressure, etc. The system of equations must be completed by the equation of state, $p=p(\rho,\ldots)$ or by the equation of the energy conservation. Consider the second case. The first law of thermodynamics $dU=-pd(\rho^{-1})+\delta Q$ can be rewritten in the form

$$\frac{dU}{dt}=-p\frac{d}{dt}\frac{1}{\rho}+\dot{Q}. \tag{A1}$$

Here, U is the internal energy per unit mass, and $\dot{Q}$ is the heat quantity introduced into the unit mass per unit time. For $U=U(p,\rho)$, Eq. (A1) becomes

$$\left(\frac{\partial U}{\partial p}\right)_\rho\frac{dp}{dt}+\left(\frac{\partial U}{\partial \rho}\right)_p\frac{d\rho}{dt}-\frac{p}{\rho^2}\frac{d\rho}{dt}=\dot{Q}. \tag{A2}$$

It follows from here that

$$\left(\frac{\partial U}{\partial \rho}\right)_p-\frac{p}{\rho^2}=\left(\frac{\partial}{\partial\rho}\left[U+\frac{p}{\rho}\right]\right)_p=\left(\frac{\partial H}{\partial \rho}\right)_p. \tag{A3}$$

Here H is the heat function (enthalpy). We obtain thus

$$\left(\frac{\partial U}{\partial p}\right)_\rho\frac{dp}{dt}+\left(\frac{\partial H}{\partial \rho}\right)_p\frac{d\rho}{dt}=\dot{Q}. \tag{A4}$$

In terms of Jacobians, this equation is written as

$$\frac{\partial(U,\rho)}{\partial(p,\rho)}\frac{dp}{dt}+\frac{\partial(H,p)}{\partial(\rho,p)}\frac{d\rho}{dt}=\dot{Q}. \tag{A5}$$

Using Jacobian's properties,[20,25] we obtain

$$\frac{\partial(U,\rho)}{\partial(p,\rho)}\frac{\partial(\rho,p)}{\partial(H,p)}\frac{dp}{dt}+\frac{d\rho}{dt}=\frac{\partial(\rho,p)}{\partial(H,p)}\dot{Q}. \tag{A6}$$

Calculation of the Jacobians gives

$$\frac{\partial(\rho,p)}{\partial(H,p)}=\frac{\partial(\rho,p)}{\partial(T,p)}\frac{\partial(T,p)}{\partial(H,p)}=-\rho\left(-\frac{1}{\rho}\frac{\partial\rho}{\partial T}\right)_p\left[\left(\frac{\partial H}{\partial T}\right)_p\right]^{-1}$$
$$=-\rho\frac{\alpha_p}{c_p}. \tag{A7}$$

Similarly,

$$\frac{\partial(U,\rho)}{\partial(p,\rho)}\frac{\partial(\rho,p)}{\partial(H,p)}=\frac{\partial(U,\rho)}{\partial(T,\rho)}\frac{\partial(T,\rho)}{\partial(T,p)}\left[\frac{\partial(T,p)}{\partial(H,p)}\right]^{-1}$$
$$=-\frac{c_v}{c_p}\left(\frac{\partial\rho}{\partial p}\right)_T=-\frac{1}{c^2}. \tag{A8}$$

Here, $c^2=(c_p/c_v)(\partial p/\partial\rho)_T$ is the square of adiabatic sound speed, T is temperature, c_p and c_v are specific heat at constant pressure and volume, respectively, and $\alpha_p=-\rho^{-1}(\partial\rho/\partial T)_p$.

Equation (A6) is transformed to

$$-\frac{1}{c^2}\frac{dp}{dt}+\frac{d\rho}{dt}=-\frac{\alpha_p}{c_p}\rho\dot{Q} \tag{A9}$$

or, in combination with a continuity equation, is written as

$$\frac{1}{\rho c^2}\frac{dp}{dt}+\mathrm{div}\,\mathbf{v}=\frac{\alpha_p}{c_p}\dot{Q}. \tag{A10}$$

The equilibrium state is characterized by the constant value of pressure p_0 and the absence of fluid motion, $\mathbf{v}=0$. For a perturbed state, $p=p_0+p_1$ and $\mathbf{v}=\mathbf{v}_1$. In linear approximation, we obtain the set of equations

$$\partial_t\mathbf{v}_1+\frac{1}{\rho_0}\nabla p_1=\frac{\mathbf{f}}{\rho_0},$$

$$\frac{1}{\rho_0c_0^2}\partial_tp_1+\mathrm{div}\,\mathbf{v}_1=\left[\frac{\alpha_p}{c_p}\right]_0\dot{Q}. \tag{A11}$$

To derive the wave equation when force and heat sources are present, we take derivatives of the expressions with respect to time and coordinates and linearly combine them. The terms with velocity derivatives cancel each other to produce the following second-order equation:

$$\rho_0\,\mathrm{div}\left[\frac{1}{\rho_0}\nabla p_1\right]-\frac{1}{c_0^2}\partial_{tt}p_1=\rho_0\,\mathrm{div}\frac{\mathbf{f}}{\rho_0}-\partial_t\left[\frac{\alpha_p\rho_0}{c_p}\right]_0\dot{Q}. \tag{A12}$$

Wave equation (A12) (without the right part and with assumed coordinate-dependent density ρ_0) was formulated by Bergmann.[26] In this context, we have to note that if the model of barotropic fluid is chosen, $\rho=\rho(p)$, i.e., density is a function of pressure only, then the density of the equilibrium state (besides the gravitational field) can only be constant (because p_0=const in the equilibrium state) and cannot depend on coordinates. If density is a function of several thermodynamical arguments [e.g., $\rho=\rho(p,T)$ or $\rho(p,s)$], then even if p_0=const, equilibrium density can be coordinate dependent, $\rho(\mathbf{x})$, if the second argument is coordinate dependent. In such case, stationary fluctuations of density can take place when some mechanisms of energy input is present in the medium. By introducing a new field variable, namely, $p_1=\sqrt{\rho_0}P$, we can eliminate the term containing the first spatial derivative of p_1 and obtain the equation structurally close to Eq. (1).

The second example of the medium, for which the pressure evolution equation has the form of Eq. (1), is a liquid with distributed gas bubbles (cavitated liquid,[27] bubble chamber,[28] upper oceanic layer,[29] jet wake, swirls, etc.).

We assume that the characteristic length λ of the acoustical wave is very large relative to the average distance d between the bubbles and to their radii R, which are small: $\lambda \gg d \gg R$. In this case, the homogeneous approximation is valid: the liquid with gas babbles can be considered as a homogeneous (on average) medium with some effective density, pressure, and other quantities (Ackeret[30]). The density of the mixture is $\rho=\rho_l(1-X)+\rho_g X$, where ρ_l and ρ_g are, respectively, the densities of liquid and gas ($\rho_g \ll \rho_l$). Quantity X is the volume fraction of gas in liquid. We assume that $X \ll 1$, which follows from $R \ll d$. Density perturbation is then $\rho'=\rho-\rho_0 \simeq (1-X_0)\rho_l' - \rho_{0g}X' \simeq c_l^{-2}p' - \rho_{0g}X' - (\rho_0 \alpha_p T_0/c_p)s'$. Here, the equilibrium density is $\rho_0=\rho_{0l}(1-X_0)+\rho_{0g}X_0 \simeq \rho_{0l}=$const and c_l is the sound speed in pure liquid. The set of linearized (with respect to field perturbations) equations, which describe the evolution of the gas-liquid mixture, is thus

$$\rho_0 \partial_t \mathbf{v}' + \nabla p' = \mathbf{f},$$

$$\partial_t \rho' + \rho_0 \operatorname{div} \mathbf{v}' = 0,$$

$$\rho_0 T_0 \partial_t s' = Q,$$

$$\rho' = \frac{1}{c_l^2} p' - \rho_{0g} X' - \frac{\rho_0 \alpha_p T_0}{c_p} s',$$

$$X'(\mathbf{x}) = \int_0^\infty dR_0 n(R_0, \mathbf{x}) V'(R_0),$$

$$\ddot{V}' + \omega_0^2 V' + \hat{D} V' = -\frac{4\pi R_0}{\rho_0} p'. \tag{A13}$$

Here, the first two equations are the mass and momentum conservation equations (in linear approximation) for the mixture, the second is the equation of thermal conduction in liquid when conduction and viscosity have little effect on the efficiency of the sound-generating mechanism (conditions for such possibility were discussed in Ref. 6), and the fourth expression for density perturbation is written in linear approximation, too. Here, p', ρ', and s' are variations of pressure, density, and entropy (per mass unit) relative to their equilibrium values, α_p is the termal expansion coefficient, and c_p is the specific heat at constant pressure. Quantities $\mathbf{f}$ and Q characterize the effects of the applied force and thermal sources in the medium. The last equation represents the linearized version of the evolution equation of one gas bubble in the external field. Dots signify the second derivative with respect to time. Bubbles are assumed to form spherical cavities and can only pulsate. The interaction between bubbles is neglected. It means that the distance between bubbles is large, $R \ll d$. The equation of motion of one bubble is governed by Rayleigh's equation.[25,31] The small spherically symmetrical volume perturbation of the gas bubble of initial radius R_0 is $V' \simeq 4\pi R_0^2 R'$, $R'=R(t)-R_0$. The bubbles are distributed with respect to their sizes according to some local distribution function $n(R_0,\mathbf{x})$, which tends to zero when $R_0 \to 0$ and $R_0 \to \infty$. The proper frequency of spherical oscillations is $\omega_0=\sqrt{3\gamma p_0/\rho_0 R_0^2}$. Here, p_0 is the gas pressure inside the bubble. The dissipative part of the equation describing the bubble pulsation can be found from the analysis of specific mechanism of energy losses or given by phenomenological estimates.

For air bubbles in water (at the atmospheric pressure p_0 and $\gamma \simeq 1.4$), one can obtain the rough estimate $\omega_0 R_0 \simeq 20$ m/s; i.e., for the bubble of $R_0 \sim 0.1$ mm, the resonance frequency is ~ 33 kHz. For air bubbles of order of microns, surface tension μ needs to be taken into account: $\omega_0 \to \omega_0 = \sqrt{3\gamma p_0/\rho_0 R_0^2 - 2\mu/\rho_0 R_0^3}$.

In the low-frequency limit, when the characteristic frequency of waves is small, $\omega \ll \omega_0$, in the equation for V' oscillations one can neglect all terms with derivatives with respect to time; i.e., quasistatic expression is valid: $V' \simeq -4\pi R_0^3 p'/3\gamma p_0$. Then by collecting all necessary expression, we find the equation for pressure in the form

$$\Delta p' - \frac{1}{c_l^2}\left[1 + \frac{\rho_0}{\rho_{0g}}\frac{c_l^2}{c_g^2}X_0(\mathbf{x})\right]\partial_t^2 p' = \operatorname{div}\mathbf{f} - \frac{\alpha_p}{c_p}\partial_t Q. \tag{A14}$$

Here, c_l is the speed of sound in pure liquid. The presence of bubbles can radically change the speed of sound in the mixture because the correcting factor $\epsilon(\mathbf{x})=(\rho_0/\rho_{0g}) \times (c_l^2/c_g^2)X_0(\mathbf{x})$ is not small even for small concentrations of bubbles. So, for $1 \gg X_0 \gg X_{cr}=(\rho_{0g}/\rho_0)(c_g^2/c_l^2)$, the second term in brackets is prevalent. For air bubbles in water, $X_{cr} \simeq 6 \times 10^{-5} \ll 1$. If pulsation ω is comparable with ω_0, dispersive and even nonlinear effects must be taken into account.[32]

APPENDIX B: DYSON'S EQUATION IN TERMS OF FUNCTIONAL DERIVATIVES AND GREEN'S FUNCTION

We expose here a simple method of finding the Green's function based on the use of functional derivatives. First of all, we introduce the trial Green's function that satisfies

$$\Delta G_0(\mathbf{x},\mathbf{x}') + k^2 G_0(\mathbf{x},\mathbf{x}') = \delta((\mathbf{x}-\mathbf{x}')). \tag{B1}$$

For infinite space, $G_0(\mathbf{x},\mathbf{x}')=-(4\pi r)^{-1}\exp(ikr)$, where $r=|\mathbf{x}-\mathbf{x}'|$. Then we rewrite Eq. (14) in the integral form

$$G(\mathbf{x},\mathbf{z}) = G_0(\mathbf{x}-\mathbf{z}) + \int d\mathbf{y} G_0(\mathbf{x}-\mathbf{y})[-k^2\epsilon(\mathbf{y})]G(\mathbf{y},\mathbf{z}) \tag{B2}$$

and average it with respect to fluctuations,

$$\langle G(\mathbf{x},\mathbf{z})\rangle = G_0(\mathbf{x}-\mathbf{z}) + \int d\mathbf{y} G_0(\mathbf{x}-\mathbf{y})\langle[-k^2\epsilon(\mathbf{y})]G(\mathbf{y},\mathbf{z})\rangle. \tag{B3}$$

For a Gaussian homogeneous process when $\langle\epsilon\rangle=0$ and $\langle\epsilon(\mathbf{x}_1)\epsilon(\mathbf{x}_2)\rangle=B(\mathbf{x}_1-\mathbf{x}_2)$, we use the expression (see Ref. 2, Chap. 20, Appendix B)

$$\begin{aligned}\langle\epsilon(\mathbf{y})G(\mathbf{y},\mathbf{z})\rangle &= \int d\mathbf{z}_1 \langle\epsilon(\mathbf{y})\epsilon(\mathbf{z}_1)\rangle \left\langle \frac{\delta G(\mathbf{y},\mathbf{z})}{\delta\epsilon(\mathbf{z}_1)} \right\rangle \\ &= \int d\mathbf{z}_1 B(\mathbf{y}-\mathbf{z}_1)\left\langle \frac{\delta G(\mathbf{y},\mathbf{z})}{\delta\epsilon(\mathbf{z}_1)} \right\rangle. \end{aligned} \tag{B4}$$

Here, the functional derivative is defined by the expression $\delta\epsilon(\mathbf{y})/\delta\epsilon(\mathbf{z})=\delta(\mathbf{y}-\mathbf{z})$, where $\delta(\mathbf{y}-\mathbf{z})$ is a 3D-Dirac function.[33] Substituting Eq. (B4) into Eq. (B3), we obtain

$$\langle G(\mathbf{x},\mathbf{z})\rangle = G_0(\mathbf{x}-\mathbf{z}) + \int d\mathbf{y} G_0(\mathbf{x}-\mathbf{y}) \times \int d\mathbf{z}_1[-k^2]B(\mathbf{y}-\mathbf{z}_1)\left\langle \frac{\delta G(\mathbf{y},\mathbf{z})}{\delta\epsilon(\mathbf{z}_1)}\right\rangle, \qquad \text{(B5)}$$

where the sought function $\langle G\rangle$ is found via its functional derivative $\langle \delta G(\mathbf{y},\mathbf{z})/\delta\epsilon(\mathbf{z}_1)\rangle$. To find this derivative, we use Eq. (B2). The functional derivative of Eq. (B2) with respect to $\epsilon(\mathbf{z}_1)$ leads to

$$\frac{\delta G(\mathbf{y},\mathbf{z})}{\delta\epsilon(\mathbf{z}_1)} = [-k^2]G_0(\mathbf{y}-\mathbf{z}_1)G(\mathbf{z}_1,\mathbf{z}) + \int d\mathbf{z}_2 G_0(\mathbf{y}-\mathbf{z}_2)[-k^2\epsilon(\mathbf{z}_2)]\frac{\delta G(\mathbf{z}_2,\mathbf{z})}{\delta\epsilon(\mathbf{z}_1)}. \qquad \text{(B6)}$$

By averaging this equation, we find

$$\left\langle \frac{\delta G(\mathbf{y},\mathbf{z})}{\delta\epsilon(\mathbf{z}_1)}\right\rangle = [-k^2]G_0(\mathbf{y}-\mathbf{z}_1)\langle G(\mathbf{z}_1,\mathbf{z})\rangle + \int d\mathbf{z}_2 G_0(\mathbf{y}-\mathbf{z}_2)\int d\mathbf{z}_3[-k^2]B(\mathbf{z}_2-\mathbf{z}_3) \times \left\langle \frac{\delta^2 G(\mathbf{z}_2,\mathbf{z})}{\delta\epsilon(\mathbf{z}_3)\,\delta\epsilon(\mathbf{z}_1)}\right\rangle. \qquad \text{(B7)}$$

The first term on the right of Eq. (B7) does not vanish when $\epsilon\to 0$; the second term is of the order $\geqslant\epsilon^2$. The next step is to substitute the derived expression into Eq. (B5). We obtain

$$\langle G(\mathbf{x},\mathbf{z})\rangle = G_0(\mathbf{x}-\mathbf{z}) + \int d\mathbf{y} G_0(\mathbf{x}-\mathbf{y})\int d\mathbf{z}_1[-k^2]B(\mathbf{y}-\mathbf{z}_1) \times\Bigg[[-k^2]G_0(\mathbf{y}-\mathbf{z}_1)\langle G(\mathbf{z}_1,\mathbf{z})\rangle + \int d\mathbf{z}_2 G_0(\mathbf{y}-\mathbf{z}_2)\int d\mathbf{z}_3[-k^2]B(\mathbf{z}_2-\mathbf{z}_3) \times\left\langle \frac{\delta^2 G(\mathbf{z}_2,\mathbf{z})}{\delta\epsilon(\mathbf{z}_3)\,\delta\epsilon(\mathbf{z}_1)}\right\rangle\Bigg],$$

or in other words,

$$\langle G(\mathbf{x},\mathbf{z})\rangle = G_0(\mathbf{x}-\mathbf{z}) + \int d\mathbf{y}d\mathbf{z}_1 G_0(\mathbf{x}-\mathbf{y})[-k^2]^2B(\mathbf{y}-\mathbf{z}_1) \times G_0(\mathbf{y}-\mathbf{z}_1)\langle G(\mathbf{z}_1,\mathbf{z})\rangle + \int d\mathbf{y}d\mathbf{z}_1d\mathbf{z}_2d\mathbf{z}_3 G_0(\mathbf{x}-\mathbf{y}) \times[-k^2]B(\mathbf{y}-\mathbf{z}_1)\times G_0(\mathbf{y}-\mathbf{z}_2)[-k^2]B(\mathbf{z}_2-\mathbf{z}_3) \times\left\langle \frac{\delta^2 G(\mathbf{z}_2,\mathbf{z})}{\delta\epsilon(\mathbf{z}_3)\,\delta\epsilon(\mathbf{z}_1)}\right\rangle. \qquad \text{(B8)}$$

The following strategy is obvious: we calculate the second derivative of Eq. (B2), average it, substitute the obtained result into Eq. (B8), and repeat the procedure over and over for higher derivatives. It is significant that the magnitude of the first term is of order ϵ^0; i.e., it is independent of fluctuations, the second term is of order ϵ^2, and the third is of order $\geqslant\epsilon^4$. If we wish to limit ourselves to the effects of order ϵ^4, no higher, we have to keep in the expression for the second derivative only the terms that do not depend on ϵ.

The second functional derivative with respect to fluctuations of Eq. (B2) [see also Eq. (B6)] gives us

$$\frac{\delta^2 G(\mathbf{z}_2,\mathbf{z})}{\delta\epsilon(\mathbf{z}_3)\,\delta\epsilon(\mathbf{z}_1)} = [-k^2]G_0(\mathbf{z}_2-\mathbf{z}_1)\frac{\delta G(\mathbf{z}_1,\mathbf{z})}{\delta\epsilon(\mathbf{z}_3)} + [-k^2]G_0(\mathbf{z}_2-\mathbf{z}_3)\frac{\delta G(\mathbf{z}_3,\mathbf{z})}{\delta\epsilon(\mathbf{z}_1)} + \int d\mathbf{z}_4 G_0(\mathbf{z}_2-\mathbf{z}_4)[-k^2\epsilon(\mathbf{z}_4)] \times\frac{\delta^2 G(\mathbf{z}_4,\mathbf{z})}{\delta\epsilon(\mathbf{z}_3)\,\delta\epsilon(\mathbf{z}_1)}. \qquad \text{(B9)}$$

Averaged Eq. (B9) is written as

$$\left\langle \frac{\delta^2 G(\mathbf{z}_2,\mathbf{z})}{\delta\epsilon(\mathbf{z}_1)\,\delta\epsilon(\mathbf{z}_3)}\right\rangle = [-k^2]G_0(\mathbf{z}_2-\mathbf{z}_1)\left\langle \frac{\delta G(\mathbf{z}_1,\mathbf{z})}{\delta\epsilon(\mathbf{z}_3)}\right\rangle + [-k^2]G_0(\mathbf{z}_2-\mathbf{z}_3)\left\langle \frac{\delta G(\mathbf{z}_3,\mathbf{z})}{\delta\epsilon(\mathbf{z}_1)}\right\rangle + \cdots. \qquad \text{(B10)}$$

Next, we use Eq. (B7) to derive the sought expression for the second derivative, which does not contain terms dependent on fluctuations,

$$\left\langle \frac{\delta^2 G(\mathbf{z}_2,\mathbf{z})}{\delta\epsilon(\mathbf{z}_1)\,\delta\epsilon(\mathbf{z}_3)}\right\rangle \simeq [-k^2]G_0(\mathbf{z}_2-\mathbf{z}_1)[-k^2]G_0(\mathbf{z}_1-\mathbf{z}_3) \times\langle G(\mathbf{z}_3,\mathbf{z})\rangle + [-k^2]G_0(\mathbf{z}_2-\mathbf{z}_3) \times[-k^2]G_0(\mathbf{z}_3-\mathbf{z}_1)\langle G(\mathbf{z}_1,\mathbf{z})\rangle. \qquad \text{(B11)}$$

Using the derived expressions for derivatives, we finally find the sought equation,[16]

$$\langle G(\mathbf{x},\mathbf{z})\rangle = G_0(\mathbf{x},\mathbf{z}) + \int d\mathbf{x}_1d\mathbf{x}_2 G_0(\mathbf{x},\mathbf{x}_1)\Sigma(\mathbf{x}_1,\mathbf{x}_2) \times\langle G(\mathbf{x}_2,\mathbf{z})\rangle. \qquad \text{(B12)}$$

The kernel $\Sigma(\mathbf{x}_1,\mathbf{x}_2)\equiv\Sigma_{12}$ of this integral equation is the set of terms

$$\Sigma_{12} = [-k^2]^2B(\mathbf{x}_1,\mathbf{x}_2)G_0(\mathbf{x}_1,\mathbf{x}_2) + [-k^2]^4\int d\mathbf{x}_3d\mathbf{x}_4 B(\mathbf{x}_1,\mathbf{x}_3)G_0(\mathbf{x}_1,\mathbf{x}_4) \times B(\mathbf{x}_4,\mathbf{x}_2)G_0(\mathbf{x}_4,\mathbf{x}_3)G_0(\mathbf{x}_3,\mathbf{x}_2) + [-k^2]^4\int d\mathbf{x}_3d\mathbf{x}_4 B(\mathbf{x}_1,\mathbf{x}_2)G_0(\mathbf{x}_1,\mathbf{x}_4) \times B(\mathbf{x}_4,\mathbf{x}_3)G_0(\mathbf{x}_4,\mathbf{x}_3)G_0(\mathbf{x}_3,\mathbf{x}_2) + \cdots. \qquad \text{(B13)}$$

Here, only terms with orders ϵ^2 and ϵ^4 are included. G_0 denotes a "bare" propagator—that is to say, the propagator in a uniform medium. The sought quantity is $\langle G\rangle$, the Green's function for the stochastic medium, also called the "dressed" propagator. To calculate *all* diagrams would amount to solving the problem exactly, which is usually not possible. We

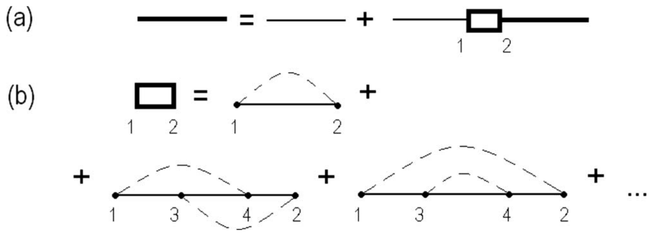

FIG. 3. Graphical (Feynman) representation of (a) Eq. (B12) and (b) Eq. (B13). Averaged Green's function is indicated by a heavy line. The thin line is the bare Green's function $G_0((\mathbf{x},\mathbf{x}_1))$. The dashed line indicates the correlation function $B(\mathbf{x},\mathbf{x}_1)$. Kernel $\Sigma(\mathbf{x},\mathbf{z})$ contains only irreducible diagrams.

therefore assume that $\langle\epsilon^2\rangle$ is small. This allows us to set up a perturbative expansion of Σ.

Equations (B12) and (B13) permit the *à la* Feynman representation (Fig. 3) if the following graphical symbols are introduced: the heavy line represents $\langle G((\mathbf{x},\mathbf{x}_1))\rangle$, the thin line corresponds to $G_0((\mathbf{x},\mathbf{x}_1))$, the thick point illustrates $[-k^2]$, and the dashed line signifies the correlation function $B(\mathbf{x},\mathbf{x}_1)$. The integration is carried out with respect to the inner variables.

Consider a randomly fluctuating in space, but statistically homogeneous medium (see, for example, Ref. 3, p. 342). In this case, statistical characteristics, such as the correlation function, do not change with translation (and rotation) of the framework, i.e., $B(\mathbf{x},\mathbf{x}_1)\equiv B(|\mathbf{x}-\mathbf{x}_1|)$. Equation (B12) becomes

$$\langle G(\mathbf{x}-\mathbf{z})\rangle = G_0(\mathbf{x}-\mathbf{z}) + \int d\mathbf{x}_1 d\mathbf{x}_2 G_0(\mathbf{x}-\mathbf{x}_1)\Sigma(\mathbf{x}_1-\mathbf{x}_2) \times\langle G(\mathbf{x}_2-\mathbf{z})\rangle. \tag{B14}$$

It is resolved via Fourier transformations defined by $Q(\mathbf{s}) = \int d\mathbf{q}(2\pi)^{-3}Q(\mathbf{q})\exp i\mathbf{q}\cdot\mathbf{s}$. After this, we multiply Eq. (B14) by $e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{z})}$ and integrate it with respect to $\mathbf{x}$ (we integrate first with respect to $\mathbf{x}_2$, then with respect to $\mathbf{x}_1$). For Fourier transforms we obtain the following equation:

$$\begin{aligned}\langle g(\mathbf{q})\rangle &= g_0(\mathbf{q}) + \int d\mathbf{x}d\mathbf{x}_1 d\mathbf{x}_2 e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}_1+\mathbf{x}_1-\mathbf{x}_2+\mathbf{x}_2-\mathbf{z})} \\ &\quad\times G_0(\mathbf{x}-\mathbf{x}_1)\Sigma(\mathbf{x}_1-\mathbf{x}_2)\langle G(\mathbf{x}_2-\mathbf{z})\rangle \\ &= g_0(\mathbf{q}) + g_0(\mathbf{q})\Sigma(\mathbf{q})\langle g(\mathbf{q})\rangle,\end{aligned} \tag{B15}$$

i.e.,

$$[1-g_0(\mathbf{q})\Sigma(\mathbf{q})]\langle g(\mathbf{q})\rangle = g_0(\mathbf{q}) \to \langle g(\mathbf{q})\rangle = [g_0^{-1}(\mathbf{q}) - \Sigma(\mathbf{q})]^{-1}, \tag{B16}$$

where[34] $g_0(\mathbf{q})=[k^2-q^2]^{-1}$. After a simple algebra, from Eq. (B16), we obtain an equation that determines where the poles are located,

$$-\langle g(\mathbf{q})\rangle^{-1} = -g_0^{-1}(\mathbf{q}) + \Sigma(\mathbf{q}) = 0. \tag{B17}$$

Applying an inverse Fourier transformation to Eq. (B16) we obtain

$$\langle G(|\mathbf{r}-\mathbf{r}_0|)\rangle = \int \frac{d\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot|\mathbf{r}-\mathbf{r}_0|} \times\left[k^2 - q^2 - \int d\mathbf{x}\Sigma(\mathbf{x})\exp(-i\mathbf{q}\cdot\mathbf{x})\right]^{-1} \tag{B18}$$

(see, for example, Ref. 3, p. 358).

APPENDIX C: RADIATION DAMPING AND CHERENKOV EFFECT

Radiation acts on bodies with certain additional force. This force is called radiation damping (or, in electrodynamics,[35] Lorentz frictional force). Let us show that the Cherenkov effect can be expressed via the work of this force.

In our case the work is characterized by the integral [see Eq. (2)]

$$\begin{aligned}\overline{\langle W_f\rangle} &= -\lim_{T\to\infty}\int_{-\infty}^{+\infty} dt\frac{d\omega}{2\pi}(-i\omega)e^{-i\omega t} \\ &\quad\times\int d\mathbf{x}\langle\phi_k(\mathbf{x})\rangle F(t)\delta(\mathbf{x}-\mathbf{x}_0(t)) \\ &= -\lim_{T\to\infty}\int_{-\infty}^{+\infty} dt\frac{d\omega}{2\pi}(-i\omega)e^{-i\omega t}\int d\mathbf{x}F(t)\delta(\mathbf{x}-\mathbf{x}_0(t)) \\ &\quad\times\int d\mathbf{x}_1\langle G_k(\mathbf{x},\mathbf{x}_1)\rangle\int_{-\infty}^{+\infty} dt_1 e^{+i\omega t_1}F(t_1)\delta(\mathbf{x}_1-\mathbf{x}_0(t_1)) \\ &= -\int_{-\infty}^{+\infty} d\omega\frac{\omega}{2\pi i}\lim_{T\to\infty}\int_{-\infty}^{+\infty} dt\int_{-\infty}^{+\infty} dt_1 e^{-i\omega(t-t_1)}F(t)F(t_1) \\ &\quad\times\langle G_k(\mathbf{x}_0(t),\mathbf{x}_0(t_1))\rangle.\end{aligned} \tag{C1}$$

Since $G_k=G_{-k}^*$, let us introduce the mixed distribution

$$\langle W(\omega,\ldots)\rangle = -\frac{\omega}{\pi}\lim_{T\to\infty}\int_{-\infty}^{+\infty} dt\int_{-\infty}^{+\infty} dt_1 F(t)F(t_1) \times\Im[e^{-i\omega(t-t_1)}\langle G_k(\mathbf{x}_0(t),\mathbf{x}_0(t_1))\rangle], \tag{C2}$$

which is normalized by the condition

$$\overline{\langle W_t\rangle} = \int_0^{\infty} d\omega\langle W(\omega,\ldots)\rangle. \tag{C3}$$

Equation (C1) shows that the entire effect is defined by the averaged Green's function $\langle G_k(\mathbf{x},\mathbf{x}_1)\rangle$. Different cases of source motion can be analyzed based on formula (C1).

Consider a source with constant productivity, $F(t)=F_0$, moving with a constant velocity, $\mathbf{x}_0(t)=\mathbf{V}t\equiv Mct$, in a fluctuating, statistically uniform, medium. In this case, the Green's function depends on the difference of arguments and has the following structure: $\langle G_k(\mathbf{x}-\mathbf{x}_1)\rangle = -(4\pi)^{-1}|\mathbf{x}-\mathbf{x}_1|^{-1}\exp{+i\Gamma_k|\mathbf{x}-\mathbf{x}_1|}$, with $\Gamma=K_k+i\gamma_k$. For low-level fluctuations $\gamma_k \ll K_k \simeq k$. After these assumptions, quantity $\langle W(\omega,\ldots)\rangle$ is

$$\langle W(\omega,\dots)\rangle = \lim_{T\to\infty}\int_{-\infty}^{+\infty} dt F_0^2 \frac{\omega}{4\pi^2}\int_{-\infty}^{+\infty} dt_1 \Im \frac{e^{-i\omega(t-t_1)+i\Gamma_k|\mathbf{V}(t-t_1)|}}{|\mathbf{V}(t-t_1)|}$$
$$= \frac{F_0^2}{4\pi^2 V}\omega\Im\int_{-\infty}^{+\infty}\frac{ds}{|s|}e^{-i(ks-\Gamma_k M|s|)} = \frac{F_0^2}{4\pi^2 M}k$$
$$\times \Im\left[\int_0^{+\infty}\frac{ds}{s}e^{-i(k-\Gamma_k M)s} + \int_0^{+\infty}\frac{ds}{s}e^{+i(k+\Gamma_k M)s}\right]. \tag{C4}$$

It is easy to see that

$$I_{\mp}(M) = \int_0^{+\infty}\frac{ds}{s}e^{(\mp ik+i\Gamma_k M)s}$$
$$= i\Gamma_k\int_{-\infty}^{M} d\mu\int_0^{+\infty} ds e^{(\mp ik+i\Gamma_k\mu)s}$$
$$= \int_{-\infty}^{M} d\mu\frac{i\Gamma_k}{\pm ik - i\Gamma_k\mu} = \int_{-\infty}^{M} d\mu\frac{\Gamma_k}{\pm k-\Gamma_k\mu}, \tag{C5}$$

i.e.,

$$I_{\mp}(M) = \int_{-\infty}^{M}\frac{d\mu}{\mu}\frac{\Gamma_k\mu \mp k}{\pm k-\Gamma_k\mu} \pm k\int_{-\infty}^{M} d\mu\frac{1}{\pm k-\Gamma_k\mu}.$$

The first (divergent) term does not contribute to the imaginary part of the expression that interests us. Therefore, the imaginary part of integral of Eq. (C4) can be described as

$$\Im\int_{-\infty}^{+\infty}\frac{ds}{|s|}e^{-i(ks-\Gamma_k M|s|)} = k\int_{-\infty}^{M} d\mu\Im\left[\frac{1}{k-K_k\mu-i\gamma_k\mu}\right.$$
$$\left.+\frac{1}{k+K_k+i\gamma_k\mu}\right]. \tag{C6}$$

For small ε and $\langle\epsilon^2\rangle \ll 1$, expression $\pi^{-1}\varepsilon/(x^2+\varepsilon^2)$ can be replaced by the Dirac function. By integrating with respect to μ, we find for the spectral power of Cherenkov radiation that

$$\langle W(\omega,\dots)\rangle \simeq \frac{F_0^2}{4\pi}\frac{k^2}{MK_k}\int_{-\infty}^{M} d\mu\,\delta^{(1)}\left(\mu-\frac{k}{K_k}\right)$$
$$\simeq \frac{F_0^2}{4\pi}\frac{k}{M}H\left(M-\frac{k}{K_k}\right). \tag{C7}$$

This expression is in agreement with Eq. (35) since $K_k \simeq k$.

[1]By the term "a source," we will describe moving objects (beams of charged or neutral particles, bodies in tenuous media, etc.) or localized regions of hydrodynamic stress transported through the medium and produced, for example, by electromagnetic field (through the release of heat or striction), moving vortices, turbulent fluxes, and so on.

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[9]For a source moving with subsonic velocity ($M=V/c<1$), the model field equation is $\Delta\phi - c^{-2}\partial_{tt}\phi = \gamma\delta^{(3)}(\mathbf{x}-\mathbf{V}t)$. The Fourier transform of the attached (intrisic) field is given by $\phi_k = (\gamma/2\pi Mc)K_0(k|\mathbf{x}_\perp|\sqrt{M^{-2}-1})\exp ikM^{-1}x_\parallel$, where $k=\omega/c$ and $x_\parallel$ is the coordinate in the source motion direction. The asymptotics of the MacDonald function are $K_0(z)\simeq -\ln z$ for $|z|\ll 1$ and $K_0(z)\simeq\sqrt{\pi/2z}\exp(-z)$ for $|z|\gg 1$; i.e., the attached intrinsic field is localized near the source.

[10]K. Yu. Platonov and G. D. Fleishman, "Transition radiation in media with random inhomogeneities," Phys. Usp. **45**, 235–291 (2002).

[11]V. Pavlov, "Transition radiation of sound in a turbulent medium," Sov. Phys. Acoust. **28**, 55–58 (1982).

[12]V. I. Pavlov and A. I. Sukhorukov, "Transition radiation of sound by a mass source moving over a rough surface," Sov. Phys. Acoust. **29**, 397–399 (1983).

[13]V. D. Lipovskii and V. V. Tamoikin, "Sound emission by moving sources in a nonuniform gaseous medium," Izv. Vyssh. Uchebn. Zaved., Radiofiz. **26**, 183–191 (1983) (in Russian).

[14]All acoustical quantities are considered here in linear approximation and are valid to the second order in ϕ.

[15]This is the standard definition of the force. See, for example, its electrodynamical analogy: V. Ginzburg, "Radiation and radiation friction force in uniformly accelerated motion of a charge," Sov. Phys. Usp. **12**, 565–574 (1970).

[16]A. B. Migdal, *Qualitative Methods in Quantum Theory*, Frontiers in Physics (Addison-Wesley, Reading, MA, 1977), Sec. 3, Chap. 5. We obtained the equation which in quantum electrodynamics is called the *Dyson equation*. Kernel Σ is called the *mass operator*.

[17]In the contemporary literature, such truncation of the series in Eq. (B13) is frequently referred to as Bourret's approximation.

[18]Near the point of phase transition, when thermodynamical fluctuations can be expressed via parameter ϵ, it would be more natural to use the correlation function of Ornstein–Zernike type $B(r)=A\langle\epsilon^2\rangle e^{-r/l}/r$. Here, $A=\kappa T\beta_T/4\pi l^2$, where κ defines the Boltzmann constant, T is the absolute temperature, and $\beta_T=\rho^{-1}(\partial_p\rho)_T$ is defined by the medium state equation $\rho=\rho(p,T)$. However, such choice of the correlation function does not change fundamentally the qualitative understanding of the process considered in the paper.

[19]H. B. Dwighte, *Tables of Integrals* (Macmillan, New York, 1961), Chap. VI: $\int dx x\ln|(1+x)/(1-x)| = 2\int dx x\ln(1+x) - \int dx x\ln|1-x^2| = (x^2-1)\ln(x+1)+x-\frac{1}{2}x^2-\frac{1}{2}[(x^2-1)\ln|x^2-1|-x^2] = x+\frac{1}{2}(x^2-1)\ln[(x+1)/(|x-1|)]$.

[20]Yu. Rumer and M. Ryvkin, *Thermodynamics, Statistical Physics and Kinetics* (Nauka, Moscow, 1977). (a) Appendix IV: $\int_0^\infty dx x^{2n}\exp(-\alpha x^2) = (2n-1)!!\sqrt{\pi}2^{-(n+1)}\alpha^{-(2n+1)/2}$. (b) Near the critical point, the fluid is sufficiently hot and compressed that the distinction between the liquid and gaseous phases is almost nonexistent. The density fluctuations in this case are strong. (c) The properties of Jacobiens are exposed in the section "Mathematical applications," p. 536.

[21]L. L. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, New York, 1984).

[22]G. Korn and T. Korn, *Mathematical Handbook for Scientists and Engineers* (McGraw-Hill, New York, 1961), Chap. 6.

[23]The complex integral $Z_0(z) = -i\pi^{-1}\int_C dt\exp(iz\cosh t)$ coincides with the Hankel function $H_0^{(1)}(z)$ for the contour, which begins at $z=-\infty+i0$, ends at $z=+\infty+i(\pi/2)$, and passes through $z=0$ (Ref. 22, 21.8-2). The MacDonald function is defined by $K_0(z)=(\pi/2)iH_0^{(1)}(iz)$. The cylindrical function Z_0 can be expanded as $Z_0(\beta|\mathbf{z}_1-\mathbf{z}_2|) = \Sigma_{k=-\infty}^{+\infty} Z_k(\beta|\mathbf{z}_1|)J_k(\beta|\mathbf{z}_2|)e^{ik(\phi_1-\phi_2)}$, where $Z_k(s)$ are the cylindrical functions of order k, $J_k(s)$ are the Bessel functions of order k, β is an arbitrary complex number, $|\mathbf{z}_1|>|\mathbf{z}_2|$, and ϕ_j are polar angles of $\mathbf{z}_j$ (see Ref. 22, 21.8-13).

[24]E. V. Pavlova and O. A. Kharin, "Acoustic transition radiation from sources crossing a chaotic phase barrier," Sov. Phys. Acoust. **38**, 496–498 (1992).

[25]L. D. Landau and E. M. Lifshitz, *Fluid Dynamics* (Pergamon, London,

1959).
[26]P. Bergmann, “The wave equation in a medium with a variable index of refraction,” J. Acoust. Soc. Am. **17**, 329–333 (1946).
[27]G. Flinn, “Physics of acoustic cavitation in liquids,” in *Physical Acoustics*, edited by W. Mason (Academic, New York, 1964), Vol. **1B**, p. 7–138.
[28]V. A. Akulichev, *Acoustic Cavitation in Cryogenic and Boiling Liquids* (Springer, The Netherlands, 1982).
[29]L. M. Brekhovskikh and Y. Lysanov, *Fundamentals of Ocean Acoustics* (Springer-Verlag, New York, 2002).
[30]J. Ackeret, “Experimentelle und theoretische Untersuchungen über Hohlraumbildung (Kavitation) im Wasser (Experimental and theoretical investigation on cavities formation (cavitation) in water),” Tech. Mech. und Thermodyn. **1**(2), 63–72 (1930).
[31]O. M. Rayleigh, “On the pressure developed in a liquid during the collapse of the spherical cavity,” Philos. Mag. **34**, 94–98 (1917).
[32]K. A. Naugol’nykh and L. A. Ostrovsky, *Nonlinear Wave Processes in Acoustics* (Nauka, Moscow, 1990).
[33]The functional derivative of functional $F[\epsilon(\mathbf{z})]=\int d\mathbf{z} f(\epsilon(\mathbf{z}))$ is calculated using $\delta F[\epsilon(z)]/\delta\epsilon(\mathbf{x})=\int d\mathbf{z}(\partial_\epsilon f(\epsilon))\delta(\epsilon(\mathbf{z}))/\delta\epsilon(\mathbf{x})$.
[34]Integrals with respect to q are calculated via contour integration by taking into account the positions of singularities, which should be contoured correctly. Therefore, one can write $[k^2-q^2]^{-1}\to[k^2-q^2+i0]^{-1}$. Introduction of symbol $+i0$ reminds us about the rule of pole-contouring. This correction is equivalent to taking into account an effective infinitesimal absorption.
[35]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Cambridge, MA, 1951).